KERALA UNIVERSITY
Model Question Paper- M. Sc. Examination
Branch : Mathematics
MM 212 - Real Analysis I
(2020 Admission onwards)

Time: 3 hours | Max. Marks:75

Part A

Answer any 5 questions from among the questions 1 to 8
Each question carries 3 marks

1. Define bounded variation of f on $[a, b]$ and discuss the bounded variation of

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), \text{if} \quad x \neq 0 \\ 0, \text{otherwise} \end{cases}$$

2. Prove that the set of all functions of bounded variation on $[a, b]$ is a linear space.
3. Assume that α is increasing on $[a, b]$. Prove that $\underline{I}(f, \alpha) \leq \overline{I}(f, \alpha)$. Give an example to show that $\underline{I}(f, \alpha) < \overline{I}(f, \alpha)$.
4. Is there exist sequence of differentiable functions $\{f_n\}$ which converges but $\{f_n'\}$ need not be convergent. Justify your answer.
5. Assume that $f_n \to f$ uniformly on S, $g_n \to g$ uniformly on S. Prove that $f_n + g_n \to f + g$ uniformly on S.
6. Find the Jacobian matrix $Df(x, y)$ for the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by the equation $f(x, y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$
7. A function can have finite directional derivative $f'(c; u)$ for every u but may fail to be continuous at c. Justify your answer.
8. Give an example to show that the second order partial derivatives $D_{1,2}f(x, y) \neq D_{2,1}f(x, y)$. $5 \times 3 = 15$

Part B

Answer all questions from 9 to 13
Each question carries 12 marks

9. .

   A. Let f be continuous on $[a, b]$. Prove that f is of bounded variation on $[a, b]$ if and only if $f = g - h$ where g and h are increasing continuous functions on $[a, b]$. 12 Marks

OR

B. (i) Let f be continuous on $[a, b]$ and f' exist and is bounded on (a, b). Prove that f is of bounded variation on $[a, b]$. Whether boundedness of f' is necessary for the bounded variation of f. Specify your answer. 5 Marks

(ii) Let f be of bounded variation on $[a, b]$ and c be an interior point of $[a, b]$. Prove that $V_f(a, b) = V_f(a, c) + V_f(c, b)$. 7 Marks

10. .

A. Define a step function and give an example. Also prove that every finite sum can be written as a Riemann-Stieltjes integral. 12 Marks

OR

B. (i) Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having end points c and d. Assume that $a = g(c)$, $b = g(d)$ and let $h(x) = f[g(x)]$, $\beta(x) = \alpha[g(x)]$, if $x \in S$. Prove that $h \in R(\beta)$ on S and that $\int_a^b f d\alpha = \int_c^d h d\beta$. 6 Marks

(ii) Assume that $f \in R(\alpha)$ on $[a, b]$ and that α has a continuous derivative α' on $[a, b]$. Prove that $\int_a^b f(x) d\alpha(x) = \int_a^b f(x)\alpha'(x) dx$. 6 Marks

11. .

A. (i) Let $\{f_n\}$ be a sequence of functions defined on a set S. Prove that there exist a function f such that $f_n \to f$ uniformly on S if and only if for every $\epsilon > 0$ there exist N such that $m, n > N$ such that $|f_m(x) - f_n(x)| < \epsilon$, for every x in S. 6 Marks

(ii) Let α be of bounded variation on $[a, b]$ and let $\{f_n\}$ be a sequence of real valued functions with $f_n \in R(\alpha)$ on $[a, b]$ for each $n = 1, 2, \ldots$ such that $f_n \to f$ uniformly on $[a, b]$. Prove that $f \in R(\alpha)$ on $[a, b]$. 6 Marks

OR

B. (i) Assume that $f_n \to f$ uniformly on S and that each f_n is continuous at a point c in S. Prove that f is continuous at c. 4 Marks

(ii) Whether the hypothesis of uniform convergence of $\{f_n\}$ in part (i) is a sufficient and necessary condition for the continuity of f ? Justify your answer. 4 Marks

(iii) Give an example of a non-uniformly convergent sequence that can integrated term by term. 4 Marks

12. .

A. Suppose that the partial derivatives $D_r f$ and $D_k f$ exist in an n-ball $B(c; \delta)$ and that both are differentiable at c. Prove that $D_{r,k} f(c) = D_{k,r} f(c)$. 12 Marks

OR

B. Assume that g is differentiable at a with total derivative $g'(a)$ and that f is differentiable at $b = g(a)$ with total derivative $f'(b)$. Prove that the composition function $f \circ g$ is differentiable at a. 12 Marks

13. .

A. State and prove the inverse function theorem. 12 Marks

OR

B. A quadric surface with center at the origin has the equation

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 1$$

Find the length of its semi-axes. 12 Marks

. $5 \times 12 = 60$